

IMSC/2004/04/18  
hep-th/0404203

# Holomorphic Factorization and Renormalization Group in Closed String Theory.

B. Sathiapalan  
*Institute of Mathematical Sciences*  
*Taramani*  
*Chennai, India 600113*  
bala@imsc.res.in

February 7, 2008

## Abstract

The prescription of Kawai, Lewellen and Tye for writing the closed string tree amplitude as sums of products of open string tree amplitudes, is applied to the world-sheet renormalization group equation. The main point is that regularization of the Minkowski (rather than Euclidean) world sheet theory allows factorization into left-moving and right-moving sectors to be maintained. Explicit calculations are done for the tachyon and the (gauge fixed) graviton.

## 1 introduction

The renormalization group approach for strings has been studied for some time by many authors [[1] - [12]]. In particular for open strings, because the calculation involves one-dimensional integrals, a lot has been done. In [9] it was shown that a proper-time equation for open strings can be written, which is essentially a Wilsonian renormalization group equation. It gives the full equation of motion, unlike the  $\beta$ -function, which is only proportional to the equation of motion. It was also shown that by keeping a finite cutoff one can go off-shell and make contact with string field theory. This was further made gauge invariant in [13] at the free level and a proposal for the gauge invariant interacting theory was made in [14, 15]. It was also subsequently generalized to include Chan-Paton factors in [16]. In order to generalize all this to closed strings we need a method that allows us to use the open string loop variable techniques.

In [17] Kawai, Lewellen and Tye (KLT) derived a prescription for writing down a closed string tree amplitude as a the sum of products of two open string amplitudes. Closed string vertex operators are products of holomorphic and anti-holomorphic vertex operators. The correlation functions therefore factorize into a product of a holomorphic function and an anti-holomorphic function. But the S-matrix amplitudes involve correlators integrated over the entire complex plane. The integration does not factorize (at least naively) and the resultant closed string amplitudes do not appear to be directly related to the open string ones. However, KLT showed that in fact, a Wick rotation into Minkowski world-sheet can be performed and what is obtained is a product of left and right moving correlation functions that are functions of two real variables ( $\sigma + \tau$  and  $\sigma - \tau$  respectively). The integrated amplitude also factorizes - except for a phase factor that retains some correlation between the two sectors. The result is a sum of terms, each of which is a product of open string amplitudes and a phase factor.

The KLT technique is for on-shell S-matrix amplitudes. We would like to get the corresponding equation of motion using the renormalization group prescription. This involves introducing a regulator and then calculating the  $\beta$ -function. The main challenge is to do this while maintaining the factorization property that KLT demonstrated for on-shell amplitudes (which does not require a regulated world-sheet theory). This is the topic of this paper.

In order to derive a renormalization group equation the first step is to regularize the theory so that there are no divergences. We do this in Minkowski world-sheet rather than in Euclidean world-sheet. The regulated

propagator in Euclidean space is  $\ln(z\bar{z} + a^2)$ . This does not factorize into a holomorphic and antiholomorphic part. In Minkowski world sheet on the other hand, the propagator can be regulated as  $\ln(\xi^2 + a^2) + \ln(\eta^2 + a^2)$ , which factorizes. This propagator is finite because  $\xi, \eta$  are real after Wick rotation. There are still infrared divergences because the variable  $\xi, \eta$  are non-compact. We will cutoff the integrals at  $\pm R$  where  $R \rightarrow \infty$ . The infrared divergences here have to be treated on the same footing as ultraviolet divergences because poles in some channels are reflected in  $\ln a$  terms, whereas poles in other channels show up as  $\ln R$  divergences. This can be achieved by setting  $R = \frac{l^2}{a}$ . Effectively we have introduced a renormalization scale  $l$ . The equations will involve the ratio  $\frac{l}{a}$ . This renormalization scheme dependence is expected in off-shell amplitudes. On shell they will disappear.

Another technique that has been used [9] is to restrict the integration region by removing small portions around the singular point. This technique cannot be used in the loop variable approach [14]. The loop variable approach requires that one should be able to define more than one vertex operator at a point. This means that the propagator itself has to be regulated. In this paper since we are more concerned with the KLT prescription as a way of possibly implementing the loop variable approach, we use the regulated propagator. But for simplicity, as will be clear later, we will also remove small portions of the contour of integration.

We will use the same techniques as KLT to derive the phase factor in the cutoff theory, i.e. we start with the Euclidean world sheet and analytically continue to Minkowski world-sheet, taking care not to cross any singularities. We assume the same propagator in the Euclidean version. The propagator is thus  $\ln(z^2 + a^2)(\bar{z}^2 + a^2)$ . While this amounts to a modification of the short distance structure, the theory is not regulated. Nevertheless in the limit  $a \rightarrow 0$  we expect to recover the correct S-matrix for on-shell states. We then analytically continue to Minkowski space, where this theory is finite. Again as  $a \rightarrow 0$  we expect to recover the S-matrix provided during the Wick rotation we do not cross any singularities. As in the  $a = 0$  case of KLT one gets a prescription for the contours and phase factors. In fact the result for the phase factors is exactly the same - it does not depend on  $a$ .

Once we have introduced a regulator, in principle the external momenta can be taken off-shell (i.e. need not satisfy the physical state conditions). For on-shell (physical) states the cutoff can be taken to zero and we recover the usual amplitudes. We can use this to obtain an R-G equation by studying the

$a$ -dependence. If the equation of motion is satisfied we expect  $d/d \ln a = 0$ . If we want the coefficient of the leading log, we can set  $l = a$  at the end. The crucial question for off-shell amplitudes is whether it can be made gauge invariant. This will not be discussed here. It is likely that the loop variable techniques used for the open string can be applied here also. This will be discussed elsewhere [18].

This paper is organized as follows. In section 2 we describe the KLT prescription as applied to the theory with a cutoff. In section 3 we apply it to get the quadratic and cubic terms in the tachyon equation (cubic and quartic terms in the Lagrangian) and also the cubic three-graviton vertex. Section 4 contains some concluding remarks.

## 2 KLT Prescription

Let us consider the integral

$$\int d^2 z |z|^{2\alpha} \quad (2.0.1)$$

We write  $z = x + iy$  and analytically continue in  $y = y' + iy''$ : Instead of integrating along the real  $y$ -axis ( $y'$ ) we continue to the imaginary axis ( $y''$ ). Thus  $z$  becomes  $x - y'' \equiv \xi$ . The old and new contours are depicted in Figure 1. Similarly  $\bar{z}$  becomes  $x + y'' \equiv \eta$ . These are of course nothing but the Minkowski space light cone coordinates - left moving and right moving respectively. The philosophy is that the Euclidean contour which is known to give the correct amplitude is continued analytically to Minkowski space, taking care not to cross any singularities. The singularities (branch points) are shown in Figure 1. They correspond to  $\xi = 0$  ('A') and  $\eta = 0$  ('B'). At A,  $y'' = x$  and therefore (at A)  $\eta = 2x$ . Thus for  $\eta > 0$  A is on the upper half plane. Similarly the point B is  $y'' = -x$ . The rotated contour C' is shown in the figure. If we draw this contour in the  $\xi$  plane we have the situation shown in Figure 2. Note that as  $y''$  increases,  $\xi$  decreases. When  $\eta > 0$  which is the case in Figure 1, as you go along the contour the branch point is on your left. This is shown in Figure 2. If  $\eta < 0$  we have the situation in Figure 3, the branch point is to the right of the contour. One can similarly draw contours for  $\eta$  by studying the branch point B. This is Figure 4,5.

From this one can deduce the phases:

$$1. \quad \xi > 0, \eta > 0 \Rightarrow \xi = |\xi|, \quad \eta = |\eta|$$

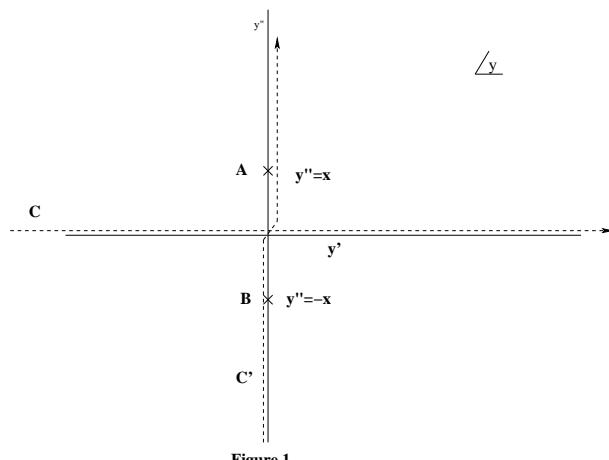


Figure 1

Figure 1: Wick rotation of contour from C to C'. The singularities at A and B are to be avoided.

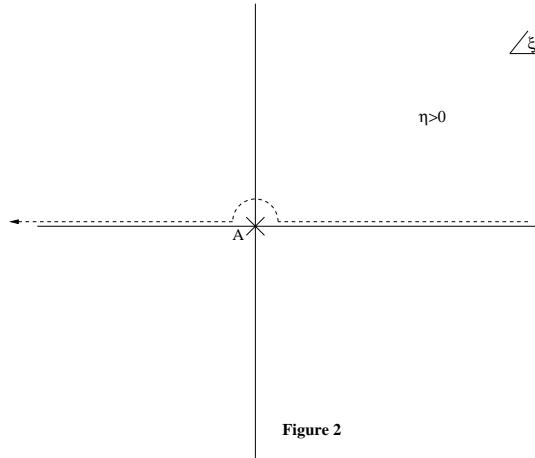


Figure 2

Figure 2: Contour in  $\xi$ -plane starts at  $+\infty$  and goes left. The singularity A is to the left of contour just as in Figure 1. This is for  $\eta > 0$ .

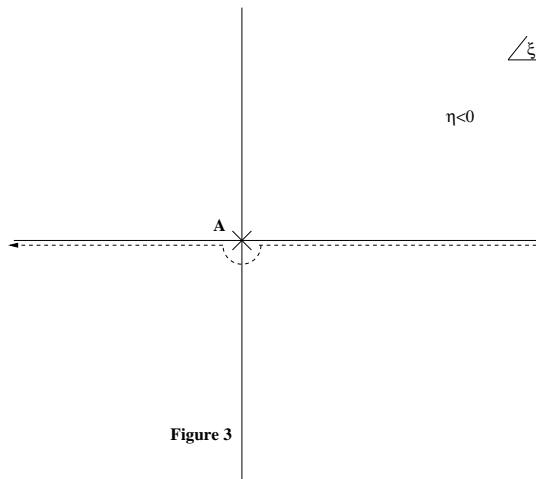


Figure 3: Contour in  $\xi$ -plane for  $\eta < 0$ .

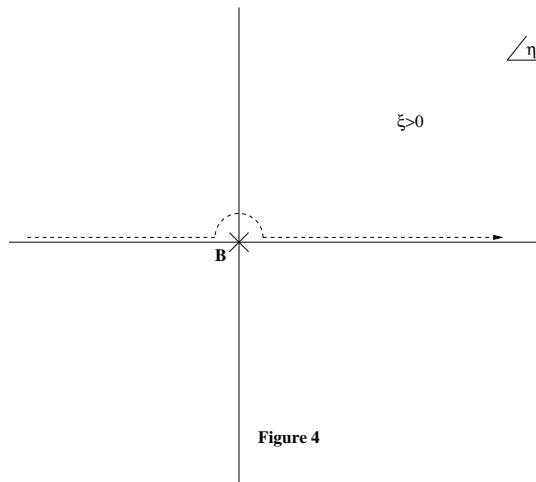


Figure 4: Contour in  $\eta$ -plane starts at  $-\infty$  and goes from left to right. For  $\xi > 0$  the singularity B is to the right of the contour.

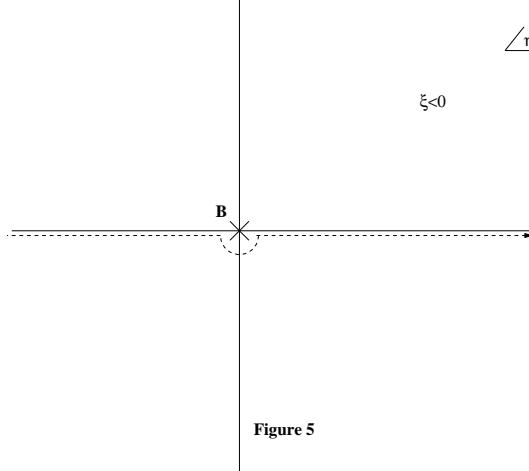


Figure 5: Contour in  $\eta$ -plane starts at  $-\infty$  and goes to the right. The singularity B is to the left of contour for  $\xi < 0$

2.  $\xi > 0, \eta < 0 \Rightarrow \xi = |\xi|, \eta = |\eta|e^{i\pi}$
3.  $\xi < 0, \eta > 0 \Rightarrow \xi = |\xi|e^{i\pi}, \eta = |\eta|$
4.  $\xi < 0, \eta < 0 \Rightarrow \xi = |\xi|e^{i\pi}, \eta = |\eta|e^{-i\pi}$  (2.0.2)

Case 4 is obtained from 3 by continuing  $\eta$ . We can also get a different phase by starting from 2 and analytically continuing  $\xi$ . However the total phase of the product  $\xi\eta$  is unaltered. It is  $\pi$  in case 2,3 and 0 in case 1,4. Thus the phase factor of  $\xi\eta$  can be expressed as  $e^{i\pi\theta(-\xi\eta)}$ .  $\theta$  is the usual step function. The integral in (2.0.1) thus becomes

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta |\xi|^{\alpha} |\eta|^{\alpha} e^{i\alpha\pi\theta(-\xi\eta)} \quad (2.0.3)$$

When there are several variables this has a simple generalization: If  $\xi_i, \xi_j, \eta_i, \eta_j$  are the variables, the phase factor becomes  $e^{i\pi\theta(-(\xi_i - \xi_j)(\eta_i - \eta_j))}$ . For each pair there is such a factor. This is the KLT prescription.

If the integral is of the form

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \xi^{\alpha_1} \eta^{\alpha_2} \quad (2.0.4)$$

with  $\alpha_1 \neq \alpha_2$ , then there is an ambiguity in the phase depending on whether we reach case 4 from case 3 or from case 2. So in this case we need a prescription. Let us assume that we follow the prescription 1-3-4, i.e.  $\xi$  is always continued before  $\eta$  in order to reach case 4. Then we get a phase  $|\xi|_1^\alpha |\eta|_2^\alpha e^{i\pi(\alpha_1 - \alpha_2)}$  for case 4. This prescription dependence will show up when we regulate the theory in order to go off-shell (see eqn (3.1.12)).

A simple regularization prescription is to cutoff the integration region around the origin. Thus we write (2.0.3) as

$$[\int_a^\infty d\xi + \int_{-\infty}^{-a} d\xi] [\int_a^\infty d\eta + \int_{-\infty}^{-a} d\eta] |\xi|^\alpha |\eta|^\alpha e^{i\alpha\pi\theta(-\xi\eta)} \quad (2.0.5)$$

If we assume that  $1 + \alpha < 0$ , we do not need an infrared regulator. We get  $2\frac{a^{2\alpha+2}}{(\alpha+1)^2} [1 - e^{i\pi(\alpha+1)}]$ . In the limit  $1 + \alpha \rightarrow 0$  this becomes  $2i\pi\frac{a^{2\epsilon}}{\epsilon}$ , where  $\epsilon = 1 + \alpha$ .

Another regularization prescription, more suited for the loop variable approach of [14] is to replace the propagator in Minkowski space by  $\ln(\xi^2 + a^2) + \ln(\eta^2 + a^2)$ . In this case the contours in the  $y$ -plane and  $\xi$ -plane are as shown in Figure 6.7. A convenient prescription is to integrate from  $+\infty$  to 0 ('a' to 'b') and 0 to  $-\infty$  ('d' to 'e'), i.e. we drop the portion 'bcd' in Figure 7. In the limit  $a \rightarrow 0$  this contribution is zero anyway. If we are concerned with off-shell amplitudes, so that  $a$  is finite, the this will modify the answers somewhat. On-shell both are equivalent. The method of regularizing the propagator has the advantage that it can easily be made gauge invariant using the loop variable approach. In the absence of further criteria to pick one off-shell prescription over another we use this one.

In this prescription (2.0.5) is replaced by

$$[\int_0^\infty d\xi + \int_{-\infty}^0 d\xi] [\int_0^\infty d\eta + \int_{-\infty}^0 d\eta] (\xi^2 + a^2)^{\frac{\alpha}{2}} (\eta^2 + a^2)^{\frac{\alpha}{2}} e^{i\alpha\pi\theta(-\xi\eta)} \quad (2.0.6)$$

The integral  $\int_0^\infty d\xi (\xi^2 + a^2)^{\frac{\alpha}{2}} = \frac{1}{2}(a)^\epsilon \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\epsilon}{2})}$ . Here, as before,  $\epsilon = 1 + \alpha$ . When  $\epsilon \rightarrow 0$  we get for the integral  $-\frac{a^\epsilon}{\epsilon}$ . The final expression for (2.0.6) in this limit is exactly the same as for (2.0.5). We will now apply all this to the tachyon and graviton.

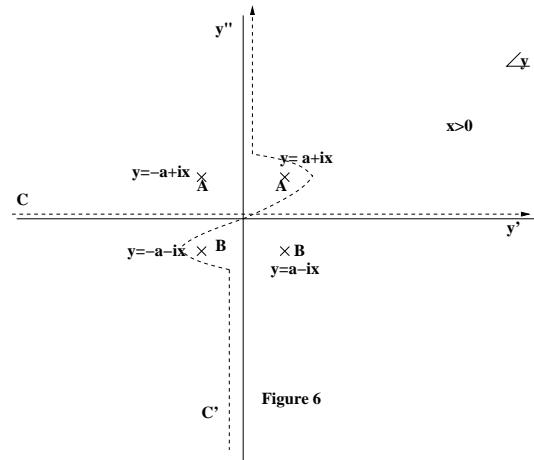


Figure 6

Figure 6: Contour  $C$  in  $y$ -plane is rotated to  $C'$  avoiding singularities  $A$  and  $B$ . The singularity  $A$  is to the left of contour for  $x > 0$ .

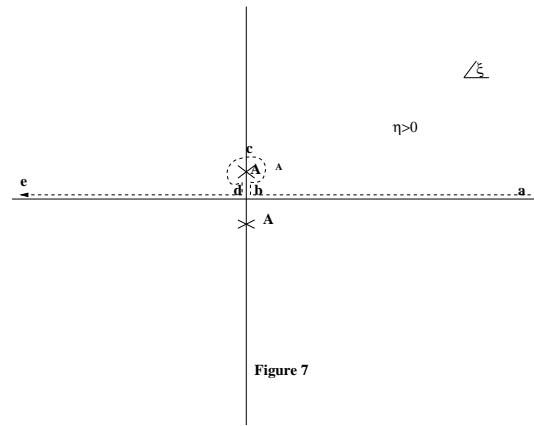


Figure 7

Figure 7: Contour  $C$  in  $\xi$ -plane. The singularity  $A$  is to the left of contour for  $\eta > 0$ . The regularization prescription is to drop the contribution of section 'bcd' of the contour.

### 3 $\beta$ -function

#### 3.1 Tachyon

The tachyon vertex operator is  $\int d^2z e^{ik.X}$  and in conventions where the propagator  $\langle X^\mu(z, \bar{z})X^\nu(w, \bar{w}) \rangle = -\frac{g^{\mu\nu}}{2} \ln |z - w|$  ( $g^{00} = -1$ ), the dimension of this vertex operator is  $\frac{k^2}{4}$ . We want the dimension to be 2 and this gives the mass shell condition  $k^2 = 8$ . Equivalently if we normal order we get  $e^{ik.X} =: e^{ik.X} : a^{\frac{k^2}{4}}$ . The vertex operator is now  $\int \frac{d^2z}{a^{\frac{k^2}{4}}} : e^{ik.X} : a^{\frac{k^2}{4}}$ . We have introduced explicit powers of  $a$  to compensate for  $d^2z$ . If we require  $\frac{d}{d \ln a} = 0$  we get the equation  $k^2 = 8$ .

At the next order we have  $\frac{1}{2!} \int d^2z_1 \int d^2z_2 \langle : e^{ik_1.X(z_1)} : : e^{ik_2.X(z_2)} : \rangle$ . This boils down to the integral  $\int d^2z |z|^{\frac{k_1+k_2}{2}}$ , where we have used  $z = z_1 - z_2$ . This corresponds to (2.0.1) with  $\alpha = \frac{k_1+k_2}{4}$ . When the two incoming tachyons and the “outgoing” tachyon with momentum  $k_1 + k_2$ , are on shell, i.e.  $k_1^2 = k_2^2 = (k_1 + k_2)^2 = 8$  we have the condition  $1 + \alpha \equiv \epsilon = 0$  and using the result of (2.0.5) we get  $\frac{2\pi i a^{2\epsilon}}{\epsilon}$ . Thus when we do  $\frac{d}{d \ln a}$  we get  $4i\pi$  as the coefficient of the quadratic term in the equation of motion (the precise normalization will not concern us here).

Now consider the cubic term in the equation of motion (quartic term in the action). We will fix one vertex operator at  $z_1 = 0$ . Thus  $\xi_1 = \eta_1 = 0$ . We thus have to evaluate

$$\begin{aligned} & \int_{-R}^{+R} d\xi_3 \int_{-R}^{+R} d\xi_2 (\xi_3 - \xi_2)^{k_2 \cdot k_3} (\xi_3)^{k_3 \cdot k_1} (\xi_2)^{k_2 \cdot k_1} \\ & \int_{-R}^{+R} d\eta_3 \int_{-R}^{+R} d\eta_2 (\eta_3 - \eta_2)^{k_2 \cdot k_3} (\eta_3)^{k_3 \cdot k_1} (\eta_2)^{k_2 \cdot k_1} \\ & e^{i\pi k_1 \cdot k_2 \theta(-(\xi_1 - \xi_2)(\eta_1 - \eta_2)) + i\pi k_1 \cdot k_3 \theta(-(\xi_1 - \xi_3)(\eta_1 - \eta_3)) + i\pi k_3 \cdot k_2 \theta(-(\xi_3 - \xi_2)(\eta_3 - \eta_2))} \end{aligned} \tag{3.1.7}$$

To get the off-shell answer we have to regulate divergences. The divergences at  $\pm\infty$  are regulated by  $R$ . The other divergences can be regulated either by cutting off the integration region or using the regulated propagator. If we are concerned about gauge invariance we should use the regulated propagator. In this paper we will use the simpler prescription of cutting off the region of integration. In an appendix we compare the two regularization schemes for a typical integral of this type.<sup>1</sup> Also we will simplify our

---

<sup>1</sup>The leading terms relevant for the on-shell calculation are seen to be the same (as expected) in both schemes.

calculation by keeping our external particles close to the mass shell so that we can use all the simplifications of the on-shell calculation. This means that all terms in the equation of motion that vanish when the external fields are on-shell, are dropped. This allows us to get away with evaluating fewer contour integrals. However if we want to go off-shell using the proper-time equation, one cannot do this.

In evaluating the integrals in (3.1.7) the simplest procedure is to fix the ordering of the  $\xi$ 's and  $\eta$ 's and use the fact that whenever  $\eta_i > \eta_j$ , the  $\xi_i$  contour goes above the branch point at  $\xi_i = \xi_j$ . Now we have two possibilities: I)  $\eta_3 > 0$  and II)  $\eta_3 < 0$ .

In each case there are *a priori* four ordering possibilities:

1.  $\eta_2 > \eta_3$  and  $\eta_2 > 0$
2.  $\eta_2 < \eta_3$  and  $\eta_2 > 0$
3.  $\eta_2 > \eta_3$  and  $\eta_2 < 0$
4.  $\eta_2 < \eta_3$  and  $\eta_2 < 0$

Of course in case I ordering 2 is not possible and in case II ordering 3 is not possible.

For each ordering the contours are shown in Figure 8. (In the figure we have  $\xi_3 > 0$ .)

It is clear that contours 1 and 4 will not contribute since they can be closed without enclosing any singularities. Thus we have contour 3 for case I and contour 2 for case II. They can both be deformed to the contour shown in Figure 9.

Our strategy will be to regulate the final contour integral along the contour shown in Figure 9. This will be done by removing a small circle around the branch point. We are guaranteed that it reduces to the S-matrix calculation when the regularization is removed. The only remaining issue is whether this is gauge invariant. This requires the loop variable techniques of [14]. Note that if instead of regulating the theory after closing the contours at  $\infty$  and dropping the ones that do not enclose singularities, we were to regulate all integrals from the beginning, the answers would be different. This is because integrals that are zero in the S-matrix calculation and therefore have been dropped entirely are no longer zero - simply because small semi-circles have been removed from them and that modifies the answer from zero to non-zero. However one expects that in the on-shell limit which is the continuum limit on the world sheet, these contributions will drop out.

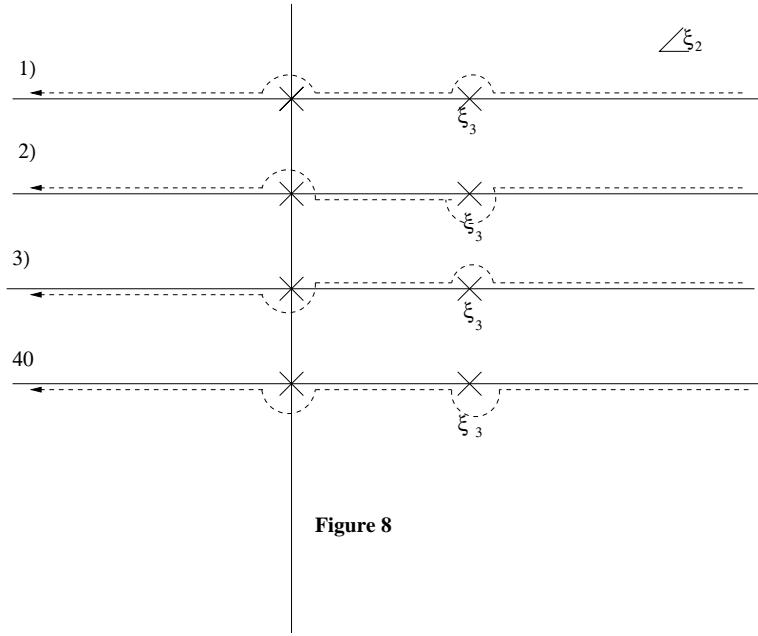


Figure 8: Contours in  $\xi_2$ -plane corresponding to the cases 1-4 listed above. The singularities are at  $\xi_2 = \xi_1 = 0$  and  $\xi_2 = \xi_3$

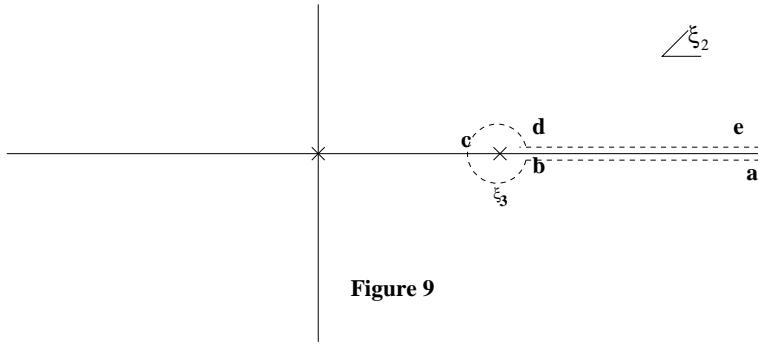


Figure 9: Contours in case 2 and 3 of figure 8 can be deformed to this contour.

Thus the off-shell equations depend on the prescription employed. It is also conceivable that one has to include all the four contours in Figure 8 when one uses the loop variable approach. This remains to be investigated.

In Figure 9 we have to evaluate  $\int_a^b + \int_d^e$ . Both integrals are equal up to a phase to

$$(\xi_3)^{k_{13}+k_{12}+k_{23}+1} [B(1+k_{32}, 1+k_{13}) - \frac{(\frac{\xi_3}{R})^{1+k_{13}}}{(1+k_{13})} - \frac{(\frac{a}{\xi_3})^{1+k_{23}}}{(1+k_{23})}] \quad (3.1.8)$$

The phases are  $e^{-i\pi k_{23}}$  and  $-e^{i\pi k_{23}}$ . They add to give a factor  $-2i \sin \pi k_{23}$ . Thus the net result is

$$(\xi_3)^{k_{13}+k_{12}+k_{23}+1} (-2i \sin \pi k_{23}) [B(1+k_{32}, 1+k_{13}) - \frac{(\frac{\xi_3}{R})^{1+k_{13}}}{(1+k_{13})} - \frac{(\frac{a}{\xi_3})^{1+k_{23}}}{(1+k_{23})}] \quad (3.1.9)$$

The regulated  $\eta_2$  integral is

$$\int_a^{\eta_3-a} d\eta_2 (\eta_3 - \eta_2)^{k_{23}} (\eta_2)^{k_{21}} (\eta_3)^{k_{31}} \quad (3.1.10)$$

This is

$$(\eta_3)^{k_{13}+k_{12}+k_{23}+1} [B(1+k_{32}, 1+k_{12}) - \frac{(\frac{a}{\eta_3})^{1+k_{12}}}{(1+k_{12})} - \frac{(\frac{a}{\eta_3})^{1+k_{23}}}{(1+k_{23})}] \quad (3.1.11)$$

The rule for  $\xi_3$  contour is the same: when  $\eta_3 < 0$ , choose the contour that goes above the singularity at  $\xi_3 = 0$ , and vice versa. The unregulated  $\xi_3$  integral would just be zero since one can close the contour at infinity. However, regularization removes the small semi-circle (of radius  $a$ ) around the origin. Thus the value of the required integral is simply the negative of the value of the integral around the small semi-circle. Thus the integrals are of the form  $\int_{-a}^{+a} d\xi_3 \int_a^R d\eta_3$ , where the integrands are given in (3.1.9,3.1.11).

There are many terms to be integrated but they are all of the type:

$$\int_{-a}^a d\xi_3 \int_a^R d\eta_3 \xi_3^{-1+\epsilon} \eta_3^{-1+\epsilon'} R^x a^{-\epsilon-\epsilon'-x} \times c \quad (3.1.12)$$

The pole term is signalled by  $\ln a$ . Thus we keep only those terms that can contribute  $a^\delta$  where  $\delta \approx 0$ . If  $\delta$  is finite and greater than zero, then this term goes to zero as  $a \rightarrow 0$ . By analyticity of the physical amplitude in momenta, we take this term to be zero for all non-zero  $\delta$ . At  $\delta = 0$

there is a pole and we are attempting to remove that. It is in this sense that we must understand the limit  $a \rightarrow 0$ . Furthermore by dimensional analysis, the final answer must be a function of  $R/a$ . Thus the coefficient of  $\ln a$  is always the same as the coefficient of  $\ln R$ . If we set  $R = \frac{l^2}{a}$  an additional factor of 2 is obtained for the coefficient of  $\ln a$ .

Note also that (3.1.12) is not of the form given in (2.0.3) because  $\epsilon$  and  $\epsilon'$  are in general not equal. Thus there is an ambiguity in the phase prescription which is  $e^{\pm i(\epsilon-\epsilon')\pi\theta(-\xi\eta)}$  (see discussion after (2.0.2)). Of course in the limit that  $\epsilon, \epsilon' \rightarrow 0$  this ambiguity disappears. This is the case in the usual  $\beta$ -function calculation where on-shell intermediate states are being removed. This is what is being attempted in this paper. However if we are far off-shell, and with a finite cutoff this ambiguity has to be dealt with by adopting a prescription. One such prescription was given in the paragraph after (2.0.4). Perhaps fortuitously, in the loop variable approach, because the propagators are Taylor expanded in (integral) powers of  $\eta, \xi$  we never see fractional powers of the kind we have in (3.1.9), (3.1.11). With integer powers there is no ambiguity. So we do not have this problem in the loop variable approach.

In (3.1.12) the  $\xi_3$  integral gives  $i\pi a^\epsilon$  (for  $\epsilon \approx 0$ ). The  $\eta_3$  integral gives  $\frac{(R^{\epsilon'} - a^{\epsilon'})}{\epsilon'}$ . If we extract the coefficient of  $\ln a$  in the product  $\frac{i\pi c a^\epsilon (R^{\epsilon'} - a^{\epsilon'}) a^{-\epsilon - \epsilon' - x} R^x}{\epsilon'}$  we get  $-i\pi c$ . The point to note is that it does not depend on any of the parameters  $\epsilon, \epsilon', x$ . Thus if we perform the operation  $\frac{d}{d \ln a}$  and evaluate at  $R = a$  (this extracts the  $\ln a$  part) we get the S-matrix with its pole parts removed, as the coefficient of the quartic term in the tachyon action:

$$\sin(\pi k_{23}) B(1+k_{23}, 1+k_{13}) B(1+k_{23}, 1+k_{12}) - \text{pole parts} \quad (3.1.13)$$

All this is of course exactly as in the case of the open string.

### 3.2 Graviton

Apart from the complication of extra indices the three graviton vertex calculation is the same as the tachyon calculation.<sup>2</sup> So we will be very brief. The graviton vertex operator can be written as

$$\int d^2 z \ k_1^\mu k_1^\nu : \partial_z X^\mu \partial_{\bar{z}} X^\nu e^{ik_0 \cdot X} : \quad (3.2.14)$$

---

<sup>2</sup>We are doing the gauge fixed case here. The gauge invariant case will be done using loop variables [18]

$k_1^\mu \bar{k}_1^\nu \approx h^{\mu\nu}$  represents a transverse traceless massless graviton. Thus the physical state conditions are  $k_0^2 = 0 = k_1 \cdot k_0 = \bar{k}_1 \cdot k_0 = k_1 \cdot \bar{k}_1$ .

The quadratic term in the equation of motion is obtained by evaluating

$$\int d^2z \ k_1^\mu \bar{k}_1^\nu : \partial_z X^\mu \partial_{\bar{z}} X^\nu e^{ik_0 \cdot X} : \int d^2w \ p_1^\rho \bar{p}_1^\sigma : \partial_w X^\rho \partial_{\bar{w}} X^\sigma e^{ip_0 \cdot X} : \quad (3.2.15)$$

as an OPE and extracting the part proportional to a graviton vertex operator.

We will only look at one of the many contractions possible. Keeping in mind that

$$\begin{aligned} \langle \partial_z X^\mu(z) X^\nu(w) \rangle &= -\frac{g^{\mu\nu}}{4(z-w)} \\ \langle X^\mu(z) \partial_w X^\nu(w) \rangle &= \frac{g^{\mu\nu}}{4(z-w)} \end{aligned} \quad (3.2.16)$$

one of the terms is:

$$[k_1^\mu p_1^\rho \frac{1}{16|z-w|^2} p_0^\mu \bar{k}_1^\nu \bar{p}_1^\sigma k_0^\sigma] : \partial_w X^\rho \partial_{\bar{z}} X^\nu e^{ik_0 X(z) + p_0 X(w)} : |z-w|^{\frac{k_0 \cdot p_0}{2}} \quad (3.2.17)$$

If we Taylor expand  $X(z)$  about  $X(w)$ , we get a graviton vertex operator  $\partial_w X^\rho \partial_{\bar{w}} X^\nu e^{i(k_0 + p_0) \cdot X(w)}$ . The integral is of the form (2.0.1) with  $\alpha = \frac{k_0 \cdot p_0}{4} - 1$ . Thus the result of doing the integral is  $\frac{2\pi i a^{2\epsilon}}{\epsilon}$  and  $\frac{d}{d \ln a}$  acting on it gives  $4\pi i$  as the coefficient of the leading log. As for the index structure, if we think of this as a vertex operator for a graviton field  $h^{\rho\nu} \approx q_1^\rho \bar{q}_1^\nu$  then the three graviton coupling implied by the equation of motion (3.2.17) is:

$$(q_1^\alpha k_1^\mu p_1^\rho) (\bar{q}_1^\beta \bar{k}_1^\nu \bar{p}_1^\sigma) [\eta^{\alpha\rho} p_0^\mu] [\eta^{\beta\nu} k_0^\sigma] 4\pi i \quad (3.2.18)$$

One can check that this agrees (up to overall normalization, which we are not concerned with at the moment) with results available in the literature [19]. The other terms in the three graviton coupling are related to this by symmetry. This concludes our discussion of the  $\beta$ -function calculation for the tachyon and graviton.

## 4 Conclusions

We have shown how the KLT prescription can be applied to the RG equation on the closed string world sheet. The main point of the construction is that

we regularize the Minkowski world sheet theory rather than the Euclidean one. This allows us to maintain the factorization of the amplitude into holomorphic and anti-holomorphic parts.

The main motivation for attempting this left-right factorization is that this gives the possibility of applying the loop variable techniques of [14] to make the closed string RG equation gauge invariant. We hope to report on this soon.

**Acknowledgements:** I would like to thank G. Date for useful discussions.

## A Appendix

In this appendix we compare the two regularizations of the integral:

$$\int_0^1 dx(1-x)^{q-1}x^{p-1} = B(p, q) \quad (\text{A.0.19})$$

The first one is

$$\begin{aligned} \int_a^{1-a} dx(1-x)^{q-1}x^{p-1} &= \int_0^1 dx(1-x)^{q-1}x^{p-1} - \int_0^a dx(1-x)^{q-1}x^{p-1} - \int_{1-a}^1 dx(1-x)^{q-1}x^{p-1} \\ &= B(p, q) - B_a(p, q) - B_a(q, p) \end{aligned} \quad (\text{A.0.20})$$

$B_a(p, q)$  is the incomplete Beta function with the expansion [20]

$$B_a(p, q) = \frac{a^p}{p} \left[ 1 + \frac{p(1-q)}{(p+1)}a + \frac{p(1-q)(2-q)}{(p+2)2!}a^2 + \dots \right] \quad (\text{A.0.21})$$

The second regularization is

$$\int_0^1 dx(x^2 + a^2)^{\frac{p-1}{2}}((1-x)^2 + a^2)^{\frac{q-1}{2}} \equiv B(p, q; a) \quad (\text{A.0.22})$$

We would like to expand in powers of  $a$  maintaining the duality symmetry between  $p$  and  $q$ . To this end we generalize the usual relation between Beta and Gamma functions. Consider the “regularized” Gamma function.

$$\int_0^\infty dx e^{-x}(x^2 + a^2)^{\frac{p-1}{2}} \equiv \tilde{\Gamma}(p, a) \quad (\text{A.0.23})$$

Using the change of variables  $x = x'y'$  and  $x + y = y'$ , we can write

$$\tilde{\Gamma}(p, a)\tilde{\Gamma}(q, a) = \int_0^\infty dx e^{-x}(x^2 + a^2)^{\frac{p-1}{2}} \int_0^\infty dy e^{-y}(y^2 + a^2)^{\frac{q-1}{2}}$$

$$\begin{aligned}
&= \int_0^\infty dy' y' e^{-y'} \int_0^1 dx' (y'^2 x'^2 + a^2)^{\frac{p-1}{2}} (y'^2 (1-x')^2 + a^2)^{\frac{q-1}{2}} \\
&= \int_0^\infty dy' (y')^{p+q-1} e^{-y'} \int_0^1 dx' (x'^2 + (\frac{a}{y'})^2)^{\frac{p-1}{2}} ((1-x')^2 + (\frac{a}{y'})^2)^{\frac{q-1}{2}} \\
&= \int_0^\infty dy' (y')^{p+q-1} e^{-y'} B(p, q; \frac{a}{y'}) \tag{A.0.24}
\end{aligned}$$

Thus using (A.0.24), if we have an expansion for  $\tilde{\Gamma}(p; a)$  in powers of  $a$  we can deduce the expansion of  $B(p, q; a)$ .

We write

$$\begin{aligned}
\int_0^\infty dx e^{-x} (x^2 + a^2)^{\frac{p-1}{2}} &\equiv \tilde{\Gamma}(p, a) = \tilde{\Gamma}_1(p, a) + \tilde{\Gamma}_2(p, a) \\
\tilde{\Gamma}_1(p, a) &= \int_0^{la} dx e^{-x} (x^2 + a^2)^{\frac{p-1}{2}} \\
\tilde{\Gamma}_2(p, a) &= \int_{la}^\infty dx e^{-x} (x^2 + a^2)^{\frac{p-1}{2}} \tag{A.0.25}
\end{aligned}$$

We have introduced an arbitrary parameter  $l$  ( $l > 1$ ). The exponential in  $\tilde{\Gamma}_1$  can be expanded in powers of  $x$  because the series converges uniformly in the interval  $(0, la)$ . In  $\tilde{\Gamma}_2$ ,  $(x^2 + a^2)^{\frac{p-1}{2}}$  can be expanded in powers of  $1/x$  because again the series converges uniformly for  $l > 1$ . The dependence on the arbitrary parameter  $l$  should cancel order by order in the sum  $\tilde{\Gamma}_1 + \tilde{\Gamma}_2$ . We will verify this to the order that we calculate.

$$\tilde{\Gamma}_1 = \sum_{n=0}^{\infty} \int_0^{la} \frac{(-x)^n}{n!} [x^2 + a^2]^{\frac{p-1}{2}} \tag{A.0.26}$$

Consider

$$\int_0^{la} x^n (x^2 + a^2)^{\frac{p-1}{2}} dx \tag{A.0.27}$$

The change of variable  $x = ax'$  followed by  $t = x'^2 + 1$  gives

$$\frac{a^{n+p}}{2} \int_1^{l^2+1} \frac{dt}{\sqrt{t-1}} (t-1)^{\frac{n}{2}} (t)^{\frac{p-1}{2}} \tag{A.0.28}$$

A further change  $s = \frac{1}{t}$  brings it into a standard form:

$$\frac{a^{n+p}}{2} \int_{\frac{1}{l^2+1}}^1 ds (1-s)^{\frac{n+1}{2}-1} s^{\frac{n+p}{2}-1}$$

$$= \frac{a^{n+p}}{2} [B\left(-\frac{n+p}{2}, \frac{n+1}{2}\right) - B_{\frac{1}{l^2+1}}\left(-\frac{n+p}{2}, \frac{n+1}{2}\right)] \quad (\text{A.0.29})$$

The leading  $n = 0, 1$  piece gives

$$\begin{aligned} \tilde{\Gamma}_1(p, a) &= \frac{a^p}{2} [B\left(-\frac{p}{2}, \frac{1}{2}\right) - B_{\frac{1}{l^2+1}}\left(-\frac{p}{2}, \frac{1}{2}\right)] - \frac{a^{1+p}}{2} [B\left(-\frac{1+p}{2}, 1\right)] + \dots \\ &= -\frac{a^p}{p} + \frac{a^{p+1}}{p+1} + \frac{(al)^p}{p} + \frac{(al)^p}{2l^2} \frac{p-1}{p-2} + \frac{(al)^p}{l^4} \frac{(p-1)(p-3)}{(p-4)} + \dots \quad (\text{A.0.30}) \end{aligned}$$

Here we have kept the  $l$ -dependent terms from the  $n = 0$  piece and only the  $l$ -independent piece from  $n = 1$ .

$$\begin{aligned} \tilde{\Gamma}_2 &= \int_{la}^{\infty} dx e^{-x} x^{p-1} \left[1 + \frac{a^2}{x^2}\right]^{\frac{p-1}{2}} \\ &= \int_{la}^{\infty} dx e^{-x} \left[x^{p-1} + a^2 \frac{p-1}{2} x^{p-3} + \frac{a^4}{2!} \left(\frac{p-1}{2}\right) \left(\frac{p-3}{2}\right) x^{p-5} + \dots\right] \\ &= \Gamma(p, la) + a^2 \frac{p-1}{2} \Gamma(p-2, la) + \frac{a^4}{2!} \left(\frac{p-1}{2}\right) \left(\frac{p-3}{2}\right) \Gamma(p-4, la) + \dots \end{aligned}$$

Here  $\Gamma(\alpha, x)$  is the incomplete Gamma function [20]. It is defined as

$$\Gamma(\alpha, x) = \int_x^{\infty} dt e^{-t} t^{\alpha-1}$$

It has the power series expansion:

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}$$

Thus  $\tilde{\Gamma}_2$  becomes

$$= \Gamma(p) - \frac{(al)^p}{p} - \frac{(al)^p}{2l^2} \frac{p-1}{p-2} - \frac{(al)^p}{l^4} \frac{(p-1)(p-3)}{(p-4)} + \dots \quad (\text{A.0.31})$$

Adding we see that the  $l$ -dependent terms cancel to the order they have been calculated, and the result is :

$$\tilde{\Gamma}(p, a) = \Gamma(p) - \frac{a^p}{p} + \frac{a^{p+1}}{p+1} + \dots \quad (\text{A.0.32})$$

Plugging all this into (A.0.24) we get for the “regularized” Beta function:

$$B(p, q; a) = B(p, q) - \frac{a^p}{p} - \frac{a^q}{q} + a^{p+1} \frac{(q-1)}{(p+1)} + a^{q+1} \frac{(p-1)}{(q+1)} + \dots \quad (\text{A.0.33})$$

Interestingly enough this agrees with the first regularization scheme to this order.

## References

- [1] C. Lovelace, Phys. Lett. B135,75 (1984).
- [2] C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262,593 (1985).
- [3] A. Sen, Phys. Rev. D32,2102 (1985). +
- [4] E. Fradkin and A.A. Tseytlin, Phys. Lett. B151,316 (1985).
- [5] C. Callan and Z. Gan, Nucl. Phys. B272, 647 (1987)
- [6] S. Das and B. Sathiapalan, Phys. Rev. Lett. B183,65 (1985).
- [7] J. Hughes, J. Liu and J. Polchinski, Nucl. Phys. B316 (1989).
- [8] T. Banks and E. Martinec, Nucl. Phys. B294, 733 (1987).
- [9] B. Sathiapalan, Nucl. Phys. B294,747 (1987).
- [10] V.A. Kostelecky, M.J. Perry, and R. Potting, Phys. Rev. Lett. 84 (2000) 4541, hep-th/9912243.
- [11] I.L. Buchbinder, O.M. Gitman, V.A. Krykhtin and Y.D. Pershin, hep-th/9910188, Nucl. Phys. B584 (2000) 615.
- [12] A.A. Tseytlin, Int. J. Mod. Phys. A4:4249 (1989) and references therein.
- [13] B. Sathiapalan, Nucl. Phys. B326, (1989) 376.
- [14] B. Sathiapalan, Mod. Phys. Lett. A17 (2002) 1175, hep-th/0201216.
- [15] B. Sathiapalan, Int.J.Mod.Phys. A18 (2003) 767, hep-th/0207098.
- [16] B. Sathiapalan, Mod. Phys. Lett. A19 (2004) 59, hep-th/0310128.
- [17] H. Kawai, D. C. Lewellen, and S.-H. H. Tye, Nucl. Phys. B269 (1986) 1.
- [18] B. Sathiapalan, In preparation.
- [19] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory”, Cambridge University Press (1987), Cambridge.
- [20] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York, 1965.